# Horizontal sections of connections on curves and transcendence

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ABSTRACT: Let $K$ be a number field, $\overline{X}$ be a smooth projective curve over it and $D$ be a reduced divisor on $\overline{X}$. Let $(E, \nabla)$ be a fibre bundle with connection having meromorphic poles on $D$. Let $p_1, \dots, p_s \in \overline{X}(K)$ and $X := \overline{X} \setminus \{D, p_1, \dots, p_s\}$ (the $p_j$'s may be in the support of $D$). Using tools from Nevanlinna theory and formal geometry, we give the definition of $E$–section of type $\alpha$ of the vector bundle $E$ with respect to the points $p_j$; this is the natural generalization of the notion of $E$ function defined in Siegel Shidlowski theory. We prove that the value of a $E$–section of type $\alpha$ in an algebraic point different from the $p_j$'s has maximal transcendence degree. Siegel Shidlowski theorem is a special case of the theorem proved. We give an application to isomonodromic connections.

## 1 Introduction.

Many questions in transcendental theory may be resumed in this "meta–question": Suppose that $U$ is a variety defined over a number field $K$ and $G(F, F^{(1)}, \dots, F^{(n)}) = 0$ is an algebraic system of differential equations defined over $U$ (the functions defining $G$ are in $K(U)$). Suppose that $F := (F_1, \dots F_n)$ is a local solution of the differential equation. Let $q \in U(K)$; what can we say about $Trdeg_{\mathbb{Q}}(K(F(q)))$? Up to the fact that this degree of transcendency is bounded from above by $Trdeg_{K(U)}(K(U)(F))$, we cannot say a lot about this question in general.

If we restrict our attention to *systems of linear differential equations* over the projective line and regular over the multiplicative group $\mathbb{G}_m$, the Siegel Shidlowski theory give us a very powerful and satisfactory answer. Let's recall the main result of the theory (in a simplified version), cf. for instance [La]:

Let

$$\frac{dY}{dz} = AY \qquad \text{with} \qquad A \in M_n(\mathbb{Q}(z)) \tag{1.1.1}$$

be a linear system of differential equations. Suppose that $F = (f_1(z), \dots, f_n(z))$ is a solution of 1.1.1 with the following properties:

(a) the functions $f_1(z), \dots, f_n(z)$ are algebraically independent over $\mathbb{C}(z)$;

(b) Each of the $f_i(z)$ has a Taylor expansion $f_i(z) = \sum_{j=0}^{\infty} a_{ij} \frac{z^j}{j!}$ with

$$a_{ij} \in \mathbb{Q} \qquad \text{and} \qquad H(a_{ij}) \ll j^{\epsilon j}$$

($H(\cdot)$ being the exponential height).

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Then, for every $q \in \mathbb{Q}^*$, we have that $Trdeg_{\mathbb{Q}}(\mathbb{Q}(f_1(q), \dots, f_n(q)) = n$. Recall that functions with property (b) above are called $E$–functions.

It is well known that the criterion above has many important consequences; in particular, the Hermite Lindeman Theorem is a special case of it (take $f_i(z) = e^{\alpha_i z}$) and non trivial transcendence properties of special values of some hypergeometric and Bessel functions can be deduced from it.

Of course, if one could generalize Siegel Shidlovski theory to arbitrary variety, the general "meta–question" above would have a satisfactory answer in the linear case. Unfortunately, the example below shows that a believable statement over an arbitrary variety is not easy to find:

**1.1 Example.** Let $f_1(z_1, z_2) = e^{z_1}$, $f_2(z_1, z_2) = e^{z_2}$ and $f_3(z_1, z_2) = z_1$; then $f_i$ are algebraically independent over $\mathbb{C}(z_1, z_2)$, they satisfy a system of linear differential equations with coefficients in $\mathbb{Q}[z_1, z_2]$ but, for every $a \in \mathbb{Q}$ the restriction of them to the line $z_1 = az_2$ are algebraically dependent. Perhaps one have to look for interactions between Siegel Shidlowski theory with the theorem of Ax [Ax].

In this paper we develop a transcendental theory, analogous to the Siegel–Shidlovski's, for horizontal sections of connections over arbitrary curves. As the example below shows, even in this case some caveat are necessary:

**1.2 Example.** consider the functions $f_1(z) = e^z$ and $f_2(z) = e^{z^2}$. They are algebraically independent, they satisfy a system of differential equation with coefficients in $\mathbb{Q}[z]$, but for every $a \in \mathbb{Q}$, we have that $f_1(a)$ and $f_2(a)$ are algebraically dependent.

This tells us that a condition analogous to (b) is necessary. Observe that an $E$–function is an entire function with order of growth one! In general, the lemma below shows that horizontal sections of vector bundles with connections are of finite growth, for a precise definition of finite order of growth, cf. §4:

**1.3 Proposition.** *Let $\Delta$ be the unit disk with coordinate $z$. Let*

$$Y' = \frac{A}{z^n} \cdot Y$$

*with $A \in \mathcal{O}_\Delta$ be a system of linear differential equations with poles only in the origin of order at most $n$. Let $F$ be an analytic solution in $\Delta^* := \Delta \setminus \{0\}$. Then*

$$\log \|F\| \leq \frac{C}{|z|^{n-1}}$$

*for a suitable positive constant $C$.*

*Proof:* The statement is standard, we give a sketch of proof for reader convenience: Let $|z| = 1$ and $t \in (0, 1)$; consider the function $h(t) := \|F(tz)\|$. Then $\frac{d}{dt}h^2(t) \leq$

$2|F(tz)F'(tz)|$, consequently $|h'(tz)| \leq \|F'(tz)\|$. This implies that there is a constant $C$ such that

$$\left|\frac{h'(t)}{h(t)}\right| \leq \frac{C}{t^n}.$$

The conclusion follows integrating both sides with respect to $t$.

On the other direction, in our paper [Ga1], as a consequence of the main results proved there, we can deduce the following:

**1.4 Theorem.** *Let $\overline{X}/_{\mathbb{Q}}$ be a smooth projective curve and $D \subseteq \overline{X}$ be a reduced divisor. Denote by $X$ the affine curve $\overline{X} \setminus D$ Let $(E; \nabla)$ be a fibre bundle with connections having meromorphic poles along $D$. Let $f_{\mathbb{C}} : X(\mathbb{C}) \to E$ be an horizontal section of finite order of growth $\rho$. Then*

$$Card(f(X(\mathbb{C})) \cap E(\mathbb{Q})) \leq \frac{rk(E)+2}{rk(E)} \cdot \rho.$$

(The theorem above is not explicitly stated in [Ga1], it can obtained as a particular case of theorem 1.1 of loc. cit.). A linear algebra manipulation may improve 1.4 to obtain:

**1.5 Corollary.** *In the hypotheses of theorem 1.4 we have that*

$$Card(f(X(\mathbb{C})) \cap E(\overline{\mathbb{Q}})) \leq \rho.$$

*Sketch of Proof*: Of course, if we apply 1.4 to the symmetric power of $E$ with the induced connection and the induced section we obtain that $Card(f(X(\mathbb{C})) \cap E(\mathbb{Q})) \leq \rho$.

Since $f$ is the horizontal section of a section, it defines a $LG$ germ of type 1 (definition in [Ga §3]) on every rational point of $X$. Let $p_1, \ldots, p_r$ be points such that $f(p_j) \in E(\overline{\mathbb{Q}})$. If we multiply a $LG$– germ of type with an integral section (over $Spec(\mathbb{Z})$) of a line bundle we obtain again an $LG$ germ and also the order of growth do not change. Thus we may suppose that the natural restriction map $H^0(E^\vee) \to \oplus_j E^\vee|_{p_j}$ is surjective. Fixing a trivialization of $E$ at each $p_j$, we see that we can find two sufficiently generic polynomials $H_j^1$ and $H_j^2$ such that $H_j^i(f|_{p_j}) = 0$. Because of the condition above, we can construct two global sections $H_i \in \oplus_{h=0}^n Sym^h(E^\vee)$, for a suitable $n$, such that $H_i|_{p_j} = H_j^i$. Consequently $F_H := (H_1(f), H_2(f))$ is an analytic section of $\mathcal{O}_X \oplus \mathcal{O}_X$ which has order of growth $\rho$, vanish at the $p_j$ and it is an $LG$–germs in these points. Due to the fact that $f$ is Zariski dense and that the $H_j^i$ are generic, one verify that $F_H$ is Zariski dense. Thus one can apply the first part of the proof and conclude.

Thus the corollary above claims that, under the condition that the order of growth is $\rho$, there are at most $\rho$ algebraic point on the image of $f$; a small variation of the argument shows that, over an arbitrary number field $K$, there can be at most $\rho[K : \mathbb{Q}]$ algebraic values. Moreover, observe that, the functions of example 1.2 are of total order two and it is not difficut to deduce from Schneider–Lang theorem that there is only one

point ($z = 0$) where the value of both functions is algebraic. Thus we see that, if an horizontal section of a vector bundle has less algebraic values on algebraic points then the order of growth, we cannot say anything about the algebraic independence of the values of the section on other algebraic points. In the limit case when the number of algebraic values is the same then the order of growth, we can say more.

This brings to the definition of $E$–section of type $\alpha$ of a vector bundle. Roughly speaking, $X$ be an affine curve defined over $\mathbb{Q}$, $V$ be vector bundle over it, $\alpha$ is an integer (this is not necessary in general but here we suppose for simplicity) and and $p_1, \dots, p_s \in X(\mathbb{Q})$; an analytic section $f : X(\mathbb{C}) \to E$ is said to be an $E$–section of type $\alpha$, if locally around each of the $p_j$ we may write $f = (f_1(z), \dots, f_m(z))$ with

$$f_i(z) = \sum_{j=0}^{\infty} a_{ij} \frac{z^j}{(j!)^\alpha} \qquad \text{with} \qquad a_{ij} \in \mathbb{Z}[\frac{1}{N}]$$

and the order of growth of $f$ is $\frac{s}{\alpha}$. The order of growth is defined using Nevanlinna theory on the Riemann surface $X(\mathbb{C})$. The precise definition is more involved and is given over an arbitrary number field and arbitrary $\alpha$, cf §6. The $E$–sections of type $\alpha$ are a good generalization of $E$–functions over arbitrary curves. Never the less it is important to notice that while the local behavior and the growth behavior of an $E$ function is resumed in its definition as a power series, the local and global properties of an $E$ sections are defined separately via formal geometry and Nevanlinna theory. In the paper [Be1] the author proves a generalization of the Schneider–Lang criterion just imposing a local Gevrey condition which is very similar to our definition of $LG$–germs. With this definition in mind we can state the main theorem proved in this paper (here, for simplicity, we state it over $\mathbb{Q}$, for the general statement cf. 7.1):

**1.6 Theorem.** *Let $\overline{X}/_{\mathbb{Q}}$ be a smooth projective curve. Let $D$ be an effective divisor on $\overline{X}$ and $(E, \nabla)$ be a fibre bundle of rank $m > 1$ with connection with meromorphic poles along $D$. Let $p_1, \dots, p_s \in \overline{X}(\mathbb{Q})$ be rational points. Let $D' := D - \{p_1, \dots, p_s\}$ and $X := \overline{X} \setminus D'$. Let $f : X(\mathbb{C}) \to E$ be an analytic* horizontal section *which is an $E$–section of type $\alpha$ with respect to the points $p_j$. Suppose that the image of $f$ is Zariski dense in $E$. Let $q \in X(\mathbb{Q})$, then*

$$Trdeg_{\mathbb{Q}}(\mathbb{Q}(f(q))) = m.$$

Observe that if $X = \mathbb{P}^1$, $D = 0 + \infty$, and we have only one point $p = 0$ we find the classical theorem by Siegel and Shidlowski. The requirement that the image is Zariski dense is equivalent to the requirement that the entries of $f$ are algebraically independent over $\overline{\mathbb{Q}}(\overline{X})$.

Even in the case when $\overline{X} = \mathbb{P}^1$ but $D$ is arbitrary, theorem 1.6 is stronger then the classical theorem by Siegel and Shidlowski. This is due to the fact that the recent papers [A1], [A2] and [Beu] prove the following: suppose we have a system of differential equations as in 1.1.1 and a solution $F := (f_1, \dots, f_n)$ such that $f_i$ are $E$–functions.

Then there is a system of differential equations $Z' = GZ$ with $G \in M_n(\mathbb{Q}[z; \frac{1}{z}])$, a solution $E$ of it and a matrix $M \in M_n(\overline{\mathbb{Q}}[z])$ such that $F = M \cdot E$. Thus if we restrict our attention to $E$–functions, or to functions which are defined by power series having similar conditions, all the transcendency results can be deduced from the classical Siegel–Shidlowki statement.

Roughly speaking the statement of 1.6 may be interpreted in the following way: Suppose that we have a solution $f$ of a linear differential equation over an affine curve and the order of growth is $\rho$. If $f$ has algebraic value on exacly $\rho$ rational points, then the values of the entries of $f$ on any other rational point are algebraically independent. Example 1.2 tells us that we cannot hope better then this.

Some words on the methodology. It is well known that there is an analogy between the arithmetic of varieties over number fields, the arithmetic of varieties over function fields and Nevanlinna theory (cf. for instance [Vo]). This analogy is a dictionary which allows to translate statements from a theory to another. In this paper (and in our previous [Ga1]) this philosophy is pushed forward: Instead of just an analogy, we try to develop an unified theory where arithmetic, analysis and algebraic geometry interact together. Usually we can find non trivial upper bounds from Nevanlinna theory and lower bounds from arithmetics, Algebraic geometry gives a common language where these two tools may interact. The papers [Be2] and [Be3] are, in our opinion, very near to the spirit of this paper; on these papers also, deep generalizations of Siegel Shidlowki theory are given.

This paper is organized as follows. In §2 we give some general criteria for an element of a complex vector space to have algebraically independent coordinates. In §3 we prove a zero lemma over arbitrary curve which replace the classical Shidlowski Lemma; the statement is formally the same then the Shidlowski Lemma, but the proof is simpler and use some tools from algebraic geometry: theory of vector bundles and Hilbert schemes. In §4 we explain the tools from Nevanlinna theory which are needed; we use a version of Nevanlinna theory (developed in [Ga1]) which allows to prove powerful lemmas of Scwhartz's type over (special kind of) Riemann surfaces. In §5 and 6 we develop the notion of $E$ sections of type $\alpha$ and explain the main properties of them. Eventually in §7 we state and prove the main theorem of the paper.

**1.7** *An application.* We can give an interesting application to the theory of the isomonodromic connections. Let $X$ be a curve and $(E_1; \nabla_1)$ and $(E_2, \nabla_2)$ two integrable connections of rank $n$ over it. Let $\rho_i : \pi_1(X) \to GL_n$ be the monodromy representation associated to $(E_i, \nabla_i)$. Suppose that $\rho_1$ is equivalent to $\rho_2$; thus the trivial representation is a subrepresentation of $\rho_1 \otimes \rho_2^\vee$; consequently we get a global horizontal section of $E_1 \otimes E_2^\vee$. Provided that it has the right order of grown, this section is the typical section to which we can apply the criterion.

We may guarantee the right order of grown by proposition 1.3. In particular it guarantees that if we have a connection on a projective curve with poles at most of order two, then an horizontal section will define an $E$–section of type one (cf. definition

6.1) over any smooth point rational point of the connection.

From this we can obtain the following: Let $X$ be a projective curve over $\mathbb{Q}$. Let $D$ be a reduced effective divisor over $X$. Denote by $U$ the affine curve $X \setminus D$. Let $(E_1; \nabla_1)$ and $(E_2, \nabla_2)$ be two fibre bundle with connection having poles at most of order two along $D$. Suppose that the corresponding representations $\rho_i : \pi_1(U) \to GL_N$ are isomorphic. Let $q \in U(\mathbb{Q})$. We can find an analytic isomorphism $\varphi : E_1 \to E_2$ over $U$, which restrict to the identity over $q$. Suppose that $Trdeg_{\mathbb{Q}(X)}(\mathbb{Q}(\varphi)) = r$

**1.7 Theorem.** *Let $V$ be an analytic neighborhood of $q$ and $p \in V \cap U(\mathbb{Q})$. Let $F$ be a local horizontal section around $q$ of $(E, \nabla_1)$. Suppose that $F(p) \in \overline{\mathbb{Q}}$, then $Trdeg_{\mathbb{Q}}(\mathbb{Q}(\varphi(F(p)) = r$.*

The proof is a direct application of Theorem 7.1.

A non trivial way to construct examples where theorem 1.7 apply, is the following: Let $B$ be a reduced divisor in $\mathbb{A}^1_{\mathbb{Q}}$. Let $X$ be a smooth projective curve defined over $\mathbb{Q}$. Let $D$ be a reduced divisor over $X$. Over $X \times \mathbb{A}^1$ consider the divisors $H_1 = D \times \mathbb{A}^1$ and $H_2 = X \times B$ and $H = H_1 + H_2$. Suppose that $(E, \nabla)$ is a fibre bundle with *integrable* connection over $X \times \mathbb{A}^1$ with poles around $H$ and which are at most of order two around $H_1$.

Then, for every $q \in \mathbb{A}^1(\mathbb{Q}) \setminus B$ the restriction $(E_q, \nabla_q)$ of $(E, \nabla)$ to $X \times \{q\}$ is a vector bundle with integrable connection having poles of order at most two along $D$.

By construction, for every couple $q_i$, $q_2 \in \mathbb{A}^1(\mathbb{Q})$, the vector bundles $(E_{q_i}, \nabla_{q_i})$ have conjugated monodromy. Thus the theorem apply in this case.

**1.9** *An explicit example.* Let $a, b, c \in \mathbb{Q}$, for every $x \in \mathbb{Q}$ consider the linear system of differential equations

$$\nabla_x : \qquad \frac{dY}{dz} = \left( \frac{1}{z^2} \cdot \begin{pmatrix} a & (a-b)x \\ 0 & b \end{pmatrix} + \frac{1}{z} \cdot \begin{pmatrix} (1-x) & -x^2 \\ 1 & 1 \end{pmatrix} + \frac{1}{z-1} \cdot \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \right) \cdot Y.$$

Then, up to conjugation, for every couple $x_0, x_1 \in \mathbb{Q}$ the linear systems $\nabla_{x_0}$ and $\nabla_{x_1}$ have the same monodromy.

*Proof:* Consider, over $\mathbb{P}^1 \times \mathbb{A}^1$, with local coordinates $(z, x)$, Denote by $\omega$ the matrix of differential forms

$$\omega := \left( \frac{1}{z^2} \cdot A(x) + \frac{1}{z} \cdot B(x) + \frac{1}{z-1} \cdot \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \right) dz + \frac{1}{x} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} dx.$$

With $A(x)$ and $B(x)$ unknown matrices to be determined. The system of differential equations

$$\mathcal{E} : \ \nabla(Y) = \omega \cdot Y$$

defines a fibre bundle with integrable connection if and only if

$$d(\omega) = \omega \wedge \omega.$$

Thus $\mathcal{E}$ is integrable if and only if $A(x)$ and $B(x)$ are solution of the linear differential system

$$\frac{dW(x)}{dx} = \frac{\left[W(x); \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right]}{x}. \tag{1.9.1}$$

A basis of solutions of the system 1.9.1 is

$$\left\{ \begin{pmatrix} 1 & \log(x) \\ 0 & 0 \end{pmatrix} ; \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} ; \begin{pmatrix} -\log(x) & -\log^2(x) \\ 1 & 0 \end{pmatrix} ; \begin{pmatrix} 0 & -\log(x) \\ 0 & 1 \end{pmatrix} \right\}$$

thus if we put $\tilde{x} = \log(x)$ and choose $\nabla_0$ to be

$$\nabla_0 : \qquad \frac{dY}{dz} = \left( \frac{1}{z^2} \cdot \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} + \frac{1}{z} \cdot \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} + \frac{1}{z-1} \cdot \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \right) \cdot Y$$

the conclusion follows.

**1.9 Remark.** One can see that the method we used above leave a lot of freedom in the choices; one can use all the powerful theory of isomonodromic deformations of connection developed for instance in [Ma]. Many cases that one may construct from this method give rise to example which, by a little of work, may deduced from the classical theorem of Siegel and Shidlowski. We were not able to reduce the transcendency problem arising from the family of differential equations proposed above from the classical theory. Unfortunately we are not able to prove that the analytic isomorphism between two different elements of the family has an high degree of transcendency over $\mathbb{Q}(z)$. In any case, it doesn't seem to us that we can obtain such an isomorphism from exponentials and algebraic functions.

## 2 Criteria for algebraic independence.

Let $K$ be a number field and $O_K$ be its ring of integers. We fix an embedding $\sigma : K \to \mathbb{C}$. Let $E$ be an hermitian $O_K$ module of rank $m$. If $V$ is an hermitian $O_K$ module, denote by $V_K$ the $K$ vector space $V \otimes_{O_K} K$ and by $V_{\mathbb{C}}$ the $\mathbb{C}$ vector space $V \otimes \mathbb{C}$ ($\mathbb{C}$ is an $O_K$ module via $\sigma$). In this section we want to describe some criteria that imply that the coordinates of an element $f \in E_{\mathbb{C}}$ are algebraically independent over $K$.

In the sequel, for every integer $n$, we denote by $E_n$ the hermitian $O_K$ module $Sym^n(E)$.

Let $f \in E_{\mathbb{C}}$. Denote by $f_n$ the image of $f^{\otimes n}$ in $(E_n)_{\mathbb{C}}$. For every $n$ denote by $V_n$ the smallest $K$–subspace containing $f_n$ and by $r_n$ its dimension.

**2.1 Definition.** *We will say that $f$ is algebraically independent over $K$ if $V_n = E_n$ for every positive integer $n$.*

**2.2 Remark.** If we fix a basis of $E_K$, $f$ is algebraically independent over $K$ if the coordinates of it are transcendental numbers algebraically independent over $\mathbb{Q}$.

We will now give some criteria which imply that $f$ is algebraically independent over $K$. First of all we observe the following trivial fact:

– Suppose that $V_1$ and $V_2$ are vector spaces and $\dim(V_2) < \dim(V_1)$ then

$$\lim_{n\to\infty} \frac{\dim(Sym^n(V_2))}{\dim(Sym^n(V_1))} = 0.$$

The proof is trivial and left to the reader.

**2.3 Lemma.** *The vector $f$ is algebraically independent over $K$ if and only if there is a constant $c > 0$ such that for every $n$,*

$$\frac{\dim(V_n)}{\dim(E_n)} \geq c.$$

*Proof:* If $f$ is algebraically independent over $K$ then, by definition we may take $c = 1$.

Conversely, suppose that $f$ is *not* algebraically independent over $K$, then there is an $n$ and a non trivial subspace $V_n \subsetneq E_n$ containing $f_n$. Thus for every integer $m$, $f_{nm} \in Sym^m(V_n) \subsetneq E_{nm}$. Consequently, there is a subsequence $n_m$ such that

$$\lim_{m\to\infty} \frac{\dim(V_{n_m})}{\dim(E_{n_m})} = 0.$$

The conclusion follows.

Observe that, the constant $c$ of the lemma above, either is zero or, a posteriori, it is one.

We will give now a criterion which implies the hypotheses of lemma 2.3. Before it we need to recall the definition of the Arakelov degree.

– If $M$ is an hermitian line bundle over $\mathrm{Spec}(O_K)$ we will define its Arakelov degree by the following formula: Let $s \in M \setminus \{0\}$; then

$$\widehat{\deg}(M) := \log(\mathrm{Card}(M/s \cdot O_K)) - \sum_{\sigma\in M_\infty} \log \|s\|_\sigma.$$

This formula is well defined because of the product formula (cf. for instance [SZ]).

– If $\overline{E}$ is an arbitrary hermitian vector bundle over $\mathrm{Spec}(O_K)$ then the line bundle $\bigwedge^{max} E$ is canonically equipped with an hermitian metric; consequently we can define the hermitian line bundle $\bigwedge^{\max} \overline{E}$. We then define $\widehat{\deg}(\overline{E}) := \deg(\bigwedge^{\max}(\overline{E}))$.

– (Cramer rule): If $E$ is an hermitian vector bundle of rank $r$, then there is a canonical hermitian isomorphism of vector bundles:

$$\det(E) \otimes E^\vee \simeq \bigwedge^{r-1} E.$$

In the following we will denote by $c_i$'s constants which are independent on the rank $m$. Since, due to lemma 2.3, in order to prove algebraic independence, we can work with symmetric products, we may always suppose that each $c_i$'s is very small compared with $m$.

We fix an hermitian line bundle $H$ over $\mathrm{Spec}(O_K)$. In the sequel, if $F$ is an hermitian $O_K$ module, for every integer $x$, we will denote $F(x)$ the hermitian vector bundle $F \otimes H^{\otimes x}$. For $P_i \in E^\vee(x)$ we denote by $F_i$ the vector $< P_i; f > \in H_{\mathbb{C}}^{\otimes x}$.

**2.4 Theorem.** *Suppose that there exist constants $c_i$ and $b_j$ (the former, possibly, depending on $m$) for which the following holds: For every $x$ sufficiently big, there exist $P_1, \ldots, P_m \in E^\vee(x)$ such that:*

– *They are linearly independent.*

– $\sup_{\sigma \in M_\infty} \{\log \|P_i\|_\sigma\} \le c_1 x \log(x) + b_1 x$.

– $\sup_{\sigma \in M_\infty} \{\log \|F_i\|_\sigma\} \le c_1 x \log(x) - c_2 \cdot m \cdot x \cdot \log(x) + c_3 \cdot x \cdot \log(x) + b_2 x$.

*Then there are constants $C_i$ depending only on the $c_i$'s (thus independent on $m$) such that*

$$r_1 \ge C_1 m + C_2.$$

*Proof:* Denote by $V_K \hookrightarrow E_K$ the minimal $K$–subspace containing $f$. Let $V := V_K \cap E$, then $r_1 = rk(V)$. For every positive integer $x$, denote by $\tilde{P}_i$ the image of $P_i$ in $V^\vee(x)$. Observe that there are constants $d_j$ such that $\widehat{\deg}(V^\vee(x)) = d_1 + d_2 \cdot x$. We can find $r_1$ elements within the $\tilde{P}_i$ which are linearly independent. without loss of generality, we may suppose that they are $\tilde{P}_1, \ldots, \tilde{P}_{r_1}$. The isomorphism of Cramer rule give rise to the following equality

$$(\tilde{P}_1 \wedge \ldots \wedge \tilde{P}_{r_1}) \otimes f = \sum_i (-1)^i (\tilde{P}_1 \wedge \ldots \wedge \hat{\tilde{P}}_i \wedge \ldots \wedge \tilde{P}_{r_1}) \otimes F_i.$$

Since $\tilde{P}_1 \wedge \ldots \wedge \tilde{P}_{r_1}$ is an *integral* section of $V^\vee(x)$, then

$$\log \|\tilde{P}_1 \wedge \ldots \wedge \tilde{P}_{r_1}\|_\sigma \ge d_1 + d_2 \cdot x - ([K:\mathbb{Q}] - 1)(r_1 \cdot c_1 \cdot x \log(x) + b_1 \cdot x).$$

Thus we find

$$\begin{aligned} & d_1 + d_2 \cdot x - ([K:\mathbb{Q}] - 1)(r_1 \cdot c_1 \cdot x \log(x) + b_1 \cdot x) + d_3 \le \\ & \le (r_1 - 1) c_1 \cdot x \log(x) + c_1 \cdot x \log(x) - m \cdot c_2 \cdot x \log(x) + c_3 \cdot x \log(x) + b_2 \cdot x + d_4; \end{aligned}$$

where the constants $b_i$'s, $c_i$'s and $d_i$'s are independent on $x$. We divide everything by $x \log(x)$ and let $x$ tend to infinity and obtain

$$r_1 \cdot c_1 [K:\mathbb{Q}] - m \cdot c_2 + c_3 \ge 0.$$

The conclusion follows.

As corollary of theorem 2.4 and lemma 2.3 we find:

**2.5 Corollary.** *Suppose that, for every $n$ sufficiently big, we may apply theorem 2.4 to $f_n$ in $(E_n)_{\mathbb{C}}$, with the $c_i$'s independent on $n$, then $f$ is algebraically independent over $K$.*

## 3 Connections and the Zero Lemma.

Let $K$ be a number field embedded in $\mathbb{C}$ and $\overline{K}$ be its algebraic closure in $\mathbb{C}$. Let $X_K$ be a smooth projective curve over $K$ and $D := \sum_i n_i P_i$ be an effective divisor on $X_K$. We will denote by $F$ the function field in one variable $\mathbb{C}(X_K)$.

If $\nabla_i : F_i \to F_i \otimes \Omega^1_{X_K}(D_i)$ $(i = 1, 2)$ are fibre bundles with singular connections on $X_K$, then the tensor product $F_1 \otimes F_2$ is naturally equipped with a singular connection $\nabla_{1,2} : F_1 \otimes F_2 \to F_1 \otimes F_2 \otimes \Omega^1_{X_K}(l.c.m.(D_1, D_2))$ (where if $D_i := \sum_j n_{i,j} P_j$, then $l.c.m.(D_1, D_2) := \sum_i \max_j\{n_{i,j}\} P_j$).

The standard derivation $d : \mathcal{O}_{X_K} \to \Omega^1_{X_K}$ induces, for every point $P \in X_K(\overline{K})$, a singular connection $\nabla^P : \mathcal{O}_{X_K}(P) \to \mathcal{O}_{X_K}(P) \otimes \Omega^1_{X_K}(P)$. Thus, for every divisor $D$ the line bundle $\mathcal{O}(D)$ is equipped with a canonical connection $\nabla^D : \mathcal{O}(D) \to \mathcal{O}(D) \otimes \Omega^1_{X_K}(|D|)$ (where, if $D := \sum_i n_i P_i$ is a divisor, we define $|D|$ to be the divisor $\sum_i \min_i\{1; |n_i|\} P_j$).

In the following we will denote by $H$ the line bundle $\mathcal{O}_{X_K}(D)$; observe that, by the construction above, $H$ is canonically equipped with a singular connection $\nabla^H : H \to H \otimes \Omega^1_{X_K}(D)$; let $s \in H^0(X_K, H)$ be a section such that $div(s) = D$. If $F$ is a coherent sheaf on $X_K$ and $x$ an integer, we will denote by $F(x)$ the sheaf $F \otimes H^{\otimes x}$.

Let $(E; \nabla^E)$ be a vector bundle on $X_K$ of rank $m$ with a singular connection

$$\nabla^E : E \longrightarrow E \otimes \Omega^1_{X_K}(D).$$

For every integer $x$, the vector bundle $E(x)$ is equipped with a singular connection $\nabla^x : E(x) \to E(x) \otimes \Omega^1_{X_K}(D)$.

Fix a point $Q \in X_K(\mathbb{C})$, which *may be in the support of* $D$. Let $\partial$ be a global section of $T_{X_K}(D)$ which do not vanish on $Q$. We fix a section $s' \in H^0(X_K; H)$ such that $s'(Q) \neq 0$. If necessary, we may enlarge $D$, thus we may suppose that such a section exists as soon as we suppose that $D$ is of sufficiently big degree. The derivation $\partial$ and the connection $\nabla^x$ induce a derivation

$$\nabla^x_\partial : E(x) \longrightarrow E(x + 2).$$

Denote by $\hat{X}_Q$ the completion of $X_K$ around $Q$ and denote by $E_Q$, $\mathbb{L}_Q$ etc. the restriction of $E$, $\mathbb{L}$ etc. to $X_Q$. We denote by $F_Q$ the completion of $F$ with respect to the discrete valuation induced by $Q$.

The restriction of $(E(x); \nabla^x)$ to the generic fibre is a $\mathcal{D}$–module $(E_F; \nabla^x)$.

**3.1 Lemma.** *Let $G \hookrightarrow E(x)$ be a sub bundle (the quotient is without torsion). The following properties are equivalent:*

*a) The bundle $G$ is a sub bundle with connection: the image of $G$ via $\nabla^x$ is contained in $G\otimes\Omega^1_{X_K}(D)$.*

*b) The $F$ vector space $G_F$ is a sub $\mathcal{D}$–module of $E(x)_F$: $\nabla(G_F)$ is contained in $(G\otimes\Omega^1_{X_K}(D))_F\subseteq(E(x)\otimes\Omega^1_{X_K}(D))_F$.*

*c) The image of the $F$ vector space $G_F$ under the map $\nabla^x_\partial$ is contained in $G(2)_F\subseteq E(x+2)_F$.*

The proof is left to the reader.

Let $P\in H^0(X_K,E(x))$; denote $P=P_0$ and $P_{i+1}:=\nabla_\partial^{x+2i}(P_i)$; fix a positive integer $r\leq m$ and denote by $\tilde{P}_i$ the elements $P_i\otimes(s')^{\otimes 2(r-i)}$. The elements $\tilde{P}_0,\dots,\tilde{P}_r$ are elements of $H^0(X_K;E(x+2r))$. Let $G\subseteq E(x+2r)$ be the sub vector bundle generated by the $\tilde{P}_i$ (the quotient is without torsion). From the lemma above we deduce

**3.2 Lemma.** *Suppose that the $\tilde{P}_i$ are linearly dependent as elements of $E(x+2r)_F$ then $G$ is a sub bundle with connection of $E(x+2r)$.*

Given a global section $P\in H^0(X_K,E(x))$, Suppose that $\tilde{P}_0,\dots,\tilde{P}_r$ are linearly independent over $F$ and $\tilde{P}_0,\dots,\tilde{P}_{r+1}$ linearly dependent; denote by $G$ the sub vector bundle generated by the $\tilde{P}_i$'s.

**3.3 Definition.** *Given a global section $P\in E(x)$, we will call the sub bundle $G\hookrightarrow E(x+2r)$ constructed above,* the minimal sub bundle with connection generated by $P$.

In particular we remark that if $\tilde{P}_0,\dots,\tilde{P}_{m-1}$ are linearly independent over $F$, then $G=E(x+2m)$.

Let $f$ be an horizontal section of $E^\vee_Q$ (the dual of $E$): namely $\nabla^{E^\vee_Q}(f)=0$.

The natural evaluation map, restricted to $f$ induces a linear map

$$\begin{aligned} ev: H^0(X_K,E(x)) &\longrightarrow H^0(X_Q,\mathcal{O}_{X_Q}(x))\\ P &\longrightarrow \langle P,f\rangle. \end{aligned}$$

In particular if $P\in H^0(X_K,E(x))$ and the minimal subbundle with connection $G$ generated by $P$ has rank $r$, we will denote by $F_i$ the sections $ev(\tilde{P}_i)\in H^0(X_Q;\mathcal{O}_{X_Q}(x+2r))$.

The main theorem of this section is the following Zero Lemma:

**3.4 Theorem.** *Suppose that we are in the hypotheses above, and that for every algebraic subbundle $K\hookrightarrow E^\vee$, we have that $f\not\in H^0(X_Q,K_Q)$. Then there exists a constant $C$ depending only on $E$, $f$ and the fixed connections, but independent on $P$ such that*

$$ord_Q(F_0)\leq x\cdot rk(G)+C.$$

Observe that $ord_Q(F_0)=ord_Q(\langle P,f\rangle)$.

**3.5 Remark.** The condition on $f$ means that $f$ is not algebraically degenerate: once one fix an algebraic trivialization of $E_F$, the coordinates of $f$ are linearly independent over $F$.

*Proof:* First of all we claim the following: $ord_Q(F_i) \geq ord_Q(F_0) - i$: by definition

$$\begin{aligned} F_i &= \langle P_i \otimes (s')^{2(r-i)}; f \rangle \\ &= \langle P_i; f \rangle \otimes (s')^{2(r-i)}; \end{aligned}$$

thus, since $s'$ do not vanish in $Q$, $ord_Q(F_i) = ord_Q(\langle P_i; f \rangle)$. Suppose that $e$ is a local generator of $H^{\otimes x+2i}$ and $z$ is a local coordinate around $Q$. Then we may suppose that $\langle P_i; f \rangle = z^a \cdot e$ for some positive integer $a$. The evaluation map

$$ev : E(x+2i) \otimes E^\vee \longrightarrow \mathcal{O}(x+2i)$$

is a morphism of vector bundles with connection; thus, we may find an analytic function $h$ in a neighborhood of Q such that

$$\begin{aligned} az^{a-1}he + z^a \nabla_\partial(e) &= \nabla_\partial \langle P_i; f \rangle \\ &= \langle \nabla_\partial P_i; f \rangle + \langle P_i; \nabla_\partial f \rangle \\ &= \langle P_{i+1}; f \rangle. \end{aligned}$$

The claim follows by induction on $i$.

We need to generalize to higher rank the notion of the order of vanishing of a section:

Let $V$ be a vector bundle on $X_K$ and $f \in H^0(X_Q; V_Q)$ be a non zero section. If we fix a trivialization of $V_Q$, we may write $f$ as $(f_1, \ldots, f_r)$ where $r$ is the rank of $V$ and $f_i$ are power series in one variable.

**3.6 Definition.** *The order of vanishing of $f$ in $Q$ is the integer* $\min_i\{ord_Q(f_i)\}$.

One easily sees that the order of vanishing of $f$ is independent on the choice of the trivialization.

The theorem will be consequence of the following lemma

**3.7 Lemma.** *There is a constant $C$ depending only on the vector bundle with connections $E$ and $f$ with the following property: let $\mathcal{F}$ be a vector bundle with connections and*

$$\alpha : E^\vee \twoheadrightarrow \mathcal{F}$$

*be a surjective morphism of vector bundles with connections. Let $[f] := \alpha(f) \in H^0(X_Q, \mathcal{F}_Q)$; then*

$$ord_Q([f]) \leq C.$$

We first show how the lemma implies the theorem. Recall the following standard properties of vector bundles:

a) (Cramer rule) If $G$ is a vector bundle of rank $r$ then there is a canonical isomorphism

$$\det(G) \otimes G^{\vee} \simeq \bigwedge^{r-1} G.$$

b) There is a constant $C$ depending only on $E$ such that, if $G \hookrightarrow E(x)$ is a sub bundle of rank $r$, then $\deg(G) \leq rx + C$.

Denote by $r$ the rank of $G$. The inclusion $G \hookrightarrow E(x+2r)$ give rise to a surjection $\alpha : E^{\vee} \twoheadrightarrow G^{\vee}(x+2r)$. Denote by $[f]$ the image of $f$ in $H^0(X_Q, G^{\vee}(x+2r))$.

We may suppose that $\tilde{P}_0, \dots, \tilde{P}_{r-1}$ are linearly independent elements of $G_F$ thus $\tilde{P}_0 \wedge \tilde{P}_1 \wedge \dots \wedge \tilde{P}_{r-1}$ is a non zero global section of $\bigwedge^r G$. Since, by property (b) above, there is a constant $C_1$ depending only on $E$ such that $\deg(\bigwedge^r G) \leq xr + C$, we have that $ord_Q(\tilde{P}_0 \wedge \tilde{P}_1 \wedge \dots \wedge \tilde{P}_{r-1}) \leq xr + C_1$. By lemma 3.7 above, there is a constant $C_2$ such that $ord_Q([f]) \leq C_2$. The isomorphism given by the Cramer rule (a) give rise to the following equality:

$$(\tilde{P}_0 \wedge \tilde{P}_1 \wedge \dots \wedge \tilde{P}_{r-1}) \otimes [f] = \sum_i (-1)^i (\tilde{P}_0 \wedge \dots \wedge \widehat{\tilde{P}_i} \wedge \dots \wedge \tilde{P}_{r-1}) \otimes F_i;$$

thus

$$C_1 + C_2 + rx \geq ord_Q((\tilde{P}_0 \wedge \tilde{P}_1 \wedge \dots \wedge \tilde{P}_{r-1}) \otimes [f]) \geq \inf_i \{ord_Q(F_i)\} \geq ord_Q(F_0) - r.$$

The conclusion follows.

**3.8 Remark.** Observe that the constant $C$ of the theorem is sum of two terms: the first is purely geometrical, it is essentially related to the measure of the stability of $E$; the second term is analytical and it is related to the structure of the specific solution of the differential equation.

*Proof:* (of Lemma 3.7) We start with a proposition:

**3.9 Proposition.** *Let $V$ be a vector bundle with singular connection on $X_K$; There exists a constant $C$ with the following property: Let $L$ be a line bundle with singular connection on $X_K$ with a surjection $\alpha : V \twoheadrightarrow L$ (as vector bundles with singular connections). Then*

$$\deg(L) \leq C.$$

We first show how the proposition implies the lemma: Apply the lemma with $V = \bigwedge^r E^{\vee}$ and we find a constant, depending only on $E$ such that, for every subbundle with connection $G$ of $E$, we have that $\deg(G^{\vee}) \leq C$.

The degrees of the sub vector bundles with connection of $E$ are uniformly bounded. Consequently, by the theory of the Hilbert scheme, we can find a projective variety $\underline{Hilb_E}$, a vector bundle $R$ on $X_K \times \underline{Hilb_E}$ and a surjection $v : pr_1^*(E) \twoheadrightarrow R$ such that, for every vector bundle $V$ with connection which is quotient of $E$, there is a point

$q \in \underline{Hilb_E}$ such that $E \twoheadrightarrow V$ is the restriction of $v$ to $X_K \times \{q\}$. For every $q \in \underline{Hilb_E}$ denote by $R_q$ the vector bundle $R|_{X_K \times \{q\}}$on $X_K$.

Let $\underline{Hilb_E}_Q$ be the completion of $X_K \times \underline{Hilb_E}$ around the Cartier divisor $\{Q\} \times \underline{Hilb_E}$. The section $f$ defines an element of $H^0(\underline{Hilb_E}_Q, R_Q)$; thus, for every $q \in \underline{Hilb_E}$, a global section $[f_q]$ of the localization $(R_q)_Q$ of $R_q$ in $Q$. Consequently we find a function

$$\begin{aligned} ord_Q : \underline{Hilb_E}(\mathbb{C}) &\longrightarrow \mathbf{Z} \\ q &\longrightarrow ord_Q([f_q]). \end{aligned}$$

The local expression of the function $ord_Q([f_q])$ shows that it is upper semi continuous for the Zariski topology and since $\underline{Hilb_E}$ is compact, the conclusion follows.

*Proof:* (of proposition 3.9) We begin by fixing some notation: denote by $m$ the rank of $V$. We fix a point $p$ on $X_K$ which is regular for the connection. Denote by $k_p$ the completion of $\mathbb{C}(X_K)$ with respect to the valuation induced by $p$. We also fix an algebraic trivialization of $V$ near $p$. Since the connection is regular around $p$, the space of horizontal sections of the module with connections $V_p$ has dimension $m$. Thus the space of *algebraic* horizontal sections of $V_p^\vee$ is finite dimensional of dimension less or equal then $m$.

Every line bundle with singular connection and quotient of $V$ defines a section $g$ of $V_p^\vee$ which is horizontal. Thus, $g$ belongs to a finite dimensional $\mathbb{C}$–vector space $W$. The line bundles $L$ which are quotient of $V$ are in bijection with points of $\mathbb{P}^{m-1}(\mathbb{C}(X_K))$ thus with algebraic maps $\varphi_L : X_K \to \mathbb{P}^{m-1}$ (modulo the action of $PGL(m)$).

Fix a basis $g_1, \ldots, g_r$ of $W$. Each $g_i$ corresponds to a quotient line bundle $L_i$ of $V$. We can associate to every line bundle with connection quotient of $V$, an element $g$ of $W$, thus a linear combination of the $g_i$'s. The lemma below shows that every line bundle quotient of $V$ which is obtained from a linear combination of the $g_i$ has degree bounded by the maximum of the degree of the $L_i$'s; thus the conclusion follows.

**3.10 Lemma.** *Let $L_i \hookrightarrow \mathcal{O}^m_{X_K}$ $(i = 1, 2)$ be sub line bundles. Consider the map*

$$\begin{aligned} + : \mathcal{O}^m_{X_K} \oplus \mathcal{O}^m_{X_K} &\longrightarrow \mathcal{O}^m_{X_K} \\ (x, y) &\longrightarrow x + y. \end{aligned}$$

*Let $M$ be the image of $L_1 \oplus L_2$ via $+$, then* $\deg(M) \geq \min\{\deg(L_i)\}$.

The proof of the lemma is elementary once one observe that there is a surjection $L_1 \oplus L_2 \twoheadrightarrow M$.

## 4 Nevanlinna theory and order of growth of sections.

In this section we will recall the main definitions and theorems about the order of growth of analytic maps. Most of these things are classical, cf. for instance [GK], but

the approach we have here is a little bit different. One may may find details and possible generalizations in [Ga].

Let $\overline{X}$ be a smooth projective curve over $\mathbb{C}$ and $D$ an effective divisor on it. Let $d$ be the degree of $D$. We denote by $X$ the affine curve $\overline{X}\setminus\{|D|\}$.

**4.1 Theorem.** *Let $p\in X$. then, up to an additive scalar, there exists a unique function $g_p:\overline{X}\to[-\infty;+\infty]$ with the following properties:*

*a) it satisfies the differential equation*

$$dd^c g_p=\delta_p-\frac{1}{d}\cdot\delta_D;$$

*$\delta_p$ (resp. $\delta_D$) being the dirac operator on $p$ (resp. on $D$).*

*b) It is a $C^\infty$ function on $\overline{X}\setminus\{p\}\cup\{|D|\}$.*

*c) There is a open neighborhood $U$ of $p$ and an harmonic function $v_p$ on $U$ such that*

$$g_p|_U=\log|z-p|^2+v_p.$$

This theorem is already proved in [Ga] in a more general situation. We give here a sketch of proof in this case for reader's convenience.

*Proof:* Fix a (Khäler) metric $\omega$ on $\overline{X}$. Let $\Delta_{\overline{\partial}}$ be the Laplace operator associated to it. The operator $T:=\delta_p-\frac{1}{d}\cdot\delta_D$ is orthogonal to the constants. Thus there is a $(1,1)$ current $\alpha$ on $\overline{X}$ such that $\Delta_{\overline{\partial}}(\alpha)=T$. Since $T$ is smooth on $\overline{X}\setminus\{p\}\cup\{|D|\}$, the form $\alpha$ is also smooth on it. The operator $L:=\cdot\wedge\omega$ induces an isomorphism between $\mathcal{D}^{(0,0)}(\overline{X})$ and $\mathcal{D}^{(1,1)}(\overline{X})$ ($\mathcal{D}^{(i,i)}(\overline{X})$ being the space of currents of degree $(i,i)$). Thus there in a function $\tilde{g}_p$ such that $L(\tilde{g}_p)=\alpha$. Since, for a suitable constant $c$, we have that $dd^c(g)=cL(\Delta_{\overline{\partial}}(g))$, points (a) and (b) are easily deduced. Point (c) is similar.

The functions $g_p$ are exhaustion functions in the sense of [GK]:

**4.2 Lemma.** *For every constant $C$, we have that $g_p^{-1}((C,+\infty])$ is a non empty neighborhood of $D$ in $\overline{X}$.*

*Proof:* Fix a metric $\|\cdot\|$ on $\mathcal{O}_{\overline{X}}(D)$. Let $\mathbb{I}$ be the canonical section of $\mathcal{O}_{\overline{X}}(D)$. By Poincaré–Lelong equation, the function $g_p+\frac{1}{d}\log\|\mathbb{I}\|^2$ is smooth near $D$. The conclusion follows.

In the following, we will call such a function $g_p$, *an exhausting function for $X$ and $p$*. Observe that an argument similar to the one above gives

**4.3 Proposition.** *Let $p$ and $q$ points on $X$. Let $g_p$ and $g_q$ be exhausting functions for $X$ and $p$ and $q$ respectively. Then there is a constant $C_{p,q}$ and an open neighborhood $V$ of $D$ such that, for every $z\in V$ we have*

$$|g_p(z)-g_q(z)|\le C_{p,q}.$$

If $p \in X$, we fix a function $g_p$ as in the theorem above. For every positive real number $r$, we consider the following two closed sets of $X$

$$B(r) := \{z \in X \text{ s.t } g_p(z) \leq \log(r)\} \quad \text{and} \quad S(r) := \{z \in X \text{ s.t } g_p(z) = \log(r)\}.$$

The function $g_p$ is strictly related with the Green function on $B(r)$:

We firstly recall the definition of the *Green functions*:

**4.4 Definition.** *Let $U$ be a regular region on a Riemann surface $M$ and $p \in U$. A* Green function *for $U$ and $p$ is a function $g_{U;p}(z)$ on $U$ such that:*

*a) $g_{U;p}(z)|_{\partial U} \equiv 0$ continuously;*

*b) $dd^c g_{U;p} = 0$ on $U \setminus \{p\}$;*

*c) near $p$, we have $g_{U;p} = -\log|z-p|^2 + \varphi$, with $\varphi$ continuous around $P$.*

One extend $g_{U,p}$ to all of $X$ by defining $g_{U,p} \equiv 0$ outside the closure of $U$. We easily deduce from the definitions that $dd^c g_{U;p} + \delta_P = \mu_{\partial U;p}$ where $\mu_{\partial U;p}$ is a positive measure of total mass one and supported on $\partial U$.

Moreover the following is true:

**4.5 Proposition.** *The Green function, if it exists, it is unique.*

The following gives the relation between the function $g_p$ and the Green functions on $B(r)$:

**4.6 Proposition.** *Let $r$ be a positive real number. The function*

$$g_p^r := \log(r) - g_p|_{B(r)}$$

*is the Green function of $B(r)$ and $p$. Consequently, for every $p$ and $q$ in $X$ there is a constant $C$, depending on $p$ and $q$, such that, for every $r$ sufficiently big,*

$$\left| g_p^r(q) - \log(r) \right| \leq C.$$

The proof follows from the definitions.

By Stokes theorem, one easily verify that, in this case, $\mu_{S(r);p}$ is the positive measure $d^c g_p|_{S(r)}$.

Let $Z$ be a projective variety and $L$ be an ample line bundle on it equipped with a positive metric. Denote by $c_1(L)$ the first Chern form associated to it.

Let $\gamma : X \to Z$ be an analytic map. We define the height function associated to it:

$$T_\gamma(r) := \int_0^r \frac{dt}{t} \int_{B(t)} \gamma^*(c_1(L)) = \int_X g_p^r \cdot \gamma^*(c_1).$$

The order of growth of the map $\gamma$ is given, as

$$\limsup_{r \to +\infty} \frac{\log T_\gamma(r)}{\log(r)}.$$

More generally, if $M$ is an hermitian line bundle on $X$, we define

$$(M,X)(r) := \int_0^r \frac{dt}{t}\int_{B(t)} c_1(M) = \int_X g_p^r \cdot c_1(M).$$

Some remarks are necessary, we can find the proofs for instance in [Ga]:

– The order of growth is independent on:

(i) The choice of the ample line bundle $L$ and on the metric on it.

(ii) The choice of the point $p$.

– If $\gamma$ is the inclusion in $\overline{X}$, or more generally if $\gamma$ is an algebraic map (cf. [GK]), then, there is a constant $C$ such that

$$\left|\frac{T_\gamma(r)}{\log(r)}\right| \leq C. \tag{4.7.1}$$

– The Stokes and Poincaré–Lelong formulas give rise to the first main theorem: Let $Y \in H^0(X,M)$ be a global section. We define *counting function* of $Y$: suppose that $div(Y) = \sum n_z z$ (the sum may possibly be infinite), and to simplify, that $p \notin div(Y)$, then

$$N_Y(r) := \int_X g_p^r \cdot \delta_{div(Y)} = \sum_{g_p^r(z)<\log(r)} n_z g_p^r(z).$$

The *First Main Theorem* (FMT) holds:

$$N_Y(r) - \int_{S(r)} \log\|Y\|^2 \mu_{S(r),p} = (M,X)(r) + \log\|Y\|^2(p).$$

The term $-\int_{S(r)} \log\|Y\|^2 \mu_{S(r),p}$ is often denoted by $m_Y(r)$ and called *proximity function of* $Y$.

Let $E \to \overline{X}$ be an hermitian vector bundle and $p : \mathbb{P} := \underline{Proj}(\mathcal{O} \oplus E^\vee) \to \overline{X}$ be the associated compactification of it. Let $\mathbb{M}$ be the tautological line bundle of $\mathbb{P}$; since $E$ is hermitian, $\mathbb{M}$ is naturally equipped with the relative Fubini–Study metric. The surjection $\mathcal{O} \oplus E^\vee \to E^\vee$ defines an inclusion $\mathbb{P}(E) \hookrightarrow \mathbb{P}$ (the divisor at infinity) and the image is a global section of $\mathbb{M}$. It is well known that, if $M$ is a sufficiently ample line bundle on $\overline{X}$ then $\mathbb{M} \otimes p^*(M)$ is a very ample line bundle on $\mathbb{P}$.

Let $f : X \to E$ be an analytic section of $E$. It canonically defines an analytic map $f_{\mathbb{P}} : X \to \mathbb{P}$. By definition, the order of growth of $f_{\mathbb{P}}$ is $\limsup_{r\to\infty} \frac{\log(f_{\mathbb{P}}^*(\mathbb{M}\otimes p^*(M));X)(r)}{\log(r)}$. Observe that by 4.7.1 the order of growth of $f_{\mathbb{P}}$ is independent on $M$.

**4.7 Definition.** *We define the order of growth of the section $f$ to be the number*

$$\rho := \limsup_{r\to\infty} \frac{\log(f_{\mathbb{P}}^*(\mathbb{M});X)(r)}{\log(r)}.$$

**4.8 Lemma.** *Suppose that $f$ is a section of order strictly less then $\rho$. Then there is a constant $C$ such that*

$$\int_{S(r)} \log \|f\| \mu_{S(r),p} \leq C r^{\rho}.$$

*Proof:* Observe that $f_{\mathbb{P}}$ do not intersect $\mathbb{P}(E)$ and that, if $q \notin \mathbb{P}(E)$, then $\|\mathbb{P}(E)\|^2(q) = \frac{1}{1+\|q\|^2}$. Thus, by FMT, there is a constant $C$ such that

$$(f_{\mathbb{P}}^*(\mathbb{M}); X)(r) = \frac{1}{2} \int_{S(r)} \log\left(1 + \|f\|^2\right) \mu_{S(r),p} + C.$$

The conclusion easily follows.

We will show that, given a section with finite order of growth, and two points, we can estimate the size of a related section in one point, if we know that this vanishes to an high order on the other point.

Fix two points $p_1$ and $p_2$ in $X$.

Suppose that $E$ is an algebraic vector bundle over $\overline{X}$. Fix an ample line bundle $H$ on $\overline{X}$. We suppose that $E$ and $H$ are equipped with smooth metrics. For every positive integer $x$, denote by $E(x)$ the vector bundle $E \otimes H^{\otimes x}$.

Fix an *analytic section* $f \in H^0(X, E^{\vee})$ having order of growth $\rho$. For every $P \in H^0(\overline{X}, E(x))$ denote by $F$ the analytic section section $f(P) \in H^0(X, H^{\otimes x})$.

We will show that one can bound the size of $F$ in $p_2$ in terms of the sup norm of $P$, the order of vanishing of $F$ in $p_1$ and the order of growth of $f$.

**4.9 Theorem.** *There is a constant $c_1$ depending only on $H$, a constant $c_2$ depending only on $f$ and a constant $c_3$ depending only on $p_1$ and $p_2$, for which the following holds:*

*For every section $P \in H^0(\overline{X}; E(x))$ such that*

*– $\log \sup\{\|P\|\} \leq B$;*

*– $ord_{p_1}(F) \geq Ax - b$;*

*We have the following estimate, for every $x \gg 0$:*

$$\log \|F\|(p_2) \leq B - \frac{Ax}{\rho} \cdot \log(x) + c_2 x + \frac{c_1}{\rho} \cdot x \log(x).$$

Observe that the constant $c_1$ depends only on $H$.

*Proof:* First of all remark that, by Cauchy–Schwartz inequality, we have that $\|F\| \leq \|P\| \cdot \|f\|$. By Stokes formula we have that, for every real number $r$,

$$\int_X \log \|F\| \cdot dd^c g g_{P_2}^r = \int_X dd^c \log \|F\| \cdot g_p^r.$$

By the definition of Green function, the Left Hand Side is

$$\int_{S(r)} \log \|F\| \mu_{S(r),p} - \log \|F\|(p_2).$$

By the Cauchy–Schwartz inequality, the hypotheses and Lemma 4.8 the first term of the sum is bounded by

$$B + c_2 r^{\rho}.$$

Since $ord_{p_1}(F) \geq Ax$, by 4.6, and the Poincaré–Lelong formula, the Right Hand Side is surely bigger then

$$(Ax - b)\log(r) - x(H, X)(r);$$

Since $H$ is algebraic, with a metric smooth at infinity, the last term of this sum is surely lower bounded by $-x \cdot c_1 \log(r)$, for a suitable $c_1$ depending only on $H$. The conclusion follows by taking $r = x^{1/\rho}$.

## 5 Order of growth at finite places.

In this section we will recall the definitions and the principal properties of the $LG$ germs. This definition is given in [Ga] and there developed in a greater generality, and here we just recall it (and explain in the special situation we need it) for reader convenience. The notion of $LG$ germ is similar to the notion of $E$ function developed by Siegel, Shidlowski and others. When we are in presence of $LG$ germs, we can estimate the order of growth of sections at all the finite places at the same time. It is our opinion that, the notion of $LG$ germs and the order of growth of sections (or more generally of analytic maps) are two concepts which may be in contrast; and from this contrast we may deduce non trivial results.

Let $K$ be a number field, $O_K$ its ring of integers and $M_K$ the set of places of $K$. We will denote by $M_{fin}$ the set of finite places of $K$. If $v \in S$,v we denote by $K_v$ the completion of $K$ with respect to $v$ and by $O_v$ its ring of integers; if $M$ is an $O_K$ module, we denote by $M_v$ the $K_v$ vector space $M \otimes K_v$ and by $M_{O_v}$ the $O_v$ module $M \otimes O_v$.

Let $X_K$ be a smooth projective curve over $K$, $H_K$ be an ample line bundle over it. Let $E_K$ be a vector bundle of rank $m$ over $X_K$. We will use the same notation then before. Denote by $E_K^{\vee}$ the dual of $E_K$

Let $\mathcal{X} \to \mathrm{Spec}(O_K)$ be a model of $X_K$. We can (and we will) suppose that $H_K$ (resp. $E_K$) extends to a line bundle $H$ (resp. to a vector bundle $E$) over $\mathcal{X}$.

Let $p_K \in X_K(K)$ be a rational point. It extends to a section $p : \mathrm{Spec}(O_K) \to \mathcal{X}$. denote by $\hat{X}_p$ the completion of $X_K$ near $p_K$. We denote $\hat{\mathcal{X}}_p$ the completion of $\mathcal{X}$ near $p$. Denote by $H_p$, $E_p$ etc (resp $H_{p,K}$, $E_{p;K}$ etc.) the restriction of $H$, $E$ etc to $\hat{\mathcal{X}}_p$ (resp. of $H_K$, $E_K$ etc to $\hat{X}_p$). Up to pass to a finite extension of $K$, we may suppose that $\hat{\mathcal{X}}_p$ is isomorphic to $Spf(O_K[\![Z]\!])$; we fix such an isomorphism; we may also suppose that $H_p$, and $E_p$ are isomorphic to the trivial bundle of the corresponding rank; we fix such isomorphisms.

Let $f \in H^0(\hat{X}_p; E_{p,K}^{\vee})$.

Since we fixed the isomorphism of $E_p^{\vee}$ with $\mathcal{O}_{\hat{\mathcal{X}}_p}^m$, the section $f$ can be written as $m$

power series. Essentially, $f$ will be a $LG$ germ, if we can control the denominators of these power series in terms of powers of the factorials.

Once we fixed the isomorphisms above, the section $f$ may be written as $m$ power series

$$f = \left( \sum_{i=1}^{\infty} a_i(1) Z^i; \ldots; \sum_{i=1}^{\infty} a_i(m) Z^i \right);$$

with $a_i(j) \in K$.

**5.1 Definition.** *We will say that $f$ is a LG–germ of type $\alpha$ if the following holds:*

*a) For every place $v \in M_K$ and every $j = 1, \ldots, m$, the power series $\sum_i a_i(j) Z^i$ have positive radius of convergence;*

*b) There is a finite set of places $S$ such that, if $v \notin S$ there is a constant $C_v$ such that, for every $j = 1, \ldots, m$,*

$$\|a_i(j)\|_v \le \frac{C_v^i}{\|i!\|_v^{\alpha}};$$

*c) $\prod_{v \notin S} C_v < \infty$.*

Following the proofs of [Ga] §3, one may prove that:

– The notion of $LG$–germ of type $\alpha$ do not depend on the chosen isomorphisms; thus the notion depends only on the germ of section. However, notice that the constants $C_v$'s may depend on the choices.

– If $E$ is equipped with a connection which is regular at $p$; a formal horizontal section is a $LG$–germ of type 1.

– If moreover, for almost all $v \in M_K$, the connection has vanishing $p$–curvature, then the formal horizontal section is an $LG$–germ of type zero.

The last two sentences are essentially proved in [Bo].

Suppose we fixed $s$ points $p_1, \ldots, p_s$ in $X_K(K)$. Suppose that, for every point $p_j$ we have an $LG$ germ $f_j \in H^0(\hat{X}_{p_j}; E^{\vee}_{p_j,K})$ of type $\alpha$. If we take a suitable blow up of $\mathcal{X}$, we may suppose that the $p_j$'s extend to sections $P_j : \mathrm{Spec}(O_K) \to \mathcal{X}$ which do not intersect.

For every integer $x$, denote by $G_x$ the $O_K$–module $H^0(\mathcal{X}; E \otimes H^{\otimes x})$. For every $j$, the The section $f_j$ induces a $O_K$–linear map $\langle \cdot ; f_j \rangle : G_x \to H^0(\hat{X}_{p_j}; H^{\otimes x}_{p_j,K})$; and, by composition a map

$$\langle \cdot ; f \rangle := (\langle \cdot ; f_1 \rangle, \langle \cdot ; f_2 \rangle, \ldots, \langle \cdot ; f_s \rangle) : G_x \longrightarrow \oplus_{j=1}^{r} H^0(\hat{X}_{p_j}; H^{\otimes x}_{p_j,K})$$

Let $j \in \{1, \ldots, s\}$. For every positive integer $i$, denote by $\hat{\mathcal{X}}^i_{p_j}$ (resp. $\hat{X}^i_{p_j}$) the $i$-th infinitesimal neighborhood of $p_j$ (resp. $p_{j,K}$) in $\mathcal{X}$ (resp. $X_K$). Similarly we denote by $H_{p_j,i}$ etc. the restriction of $H$ etc. to $\hat{\mathcal{X}}^i_{p_j}$. Since $\mathcal{X}$ is smooth near each of $p_j$'s, the sheaf of Khäler differentials $\Omega^1_{\mathcal{X}/O_K}$ is locally free in a neighborhood of $p_j$. Denote by $T_{p_j}\mathcal{X}$ the restriction to $p_j$ of the dual of it.

Denote by $res_j : \oplus_j H^0(\hat{X}_{p_j}; H^{\otimes x}_{p_j,K}) \to \oplus_j H^0(\hat{X}^i_{p_j}; H^{\otimes x}_{p_j,i}) \otimes K$ the restriction map, by $\langle \cdot ; f\rangle_i$ the map obtained by composing $\langle \cdot ; f\rangle$ with the $res_j$ and by $G^i_x$ the kernel of $\langle \cdot ; f\rangle_i$.

The exact sequence

$$0 \to \oplus_{j=1}^s H^0(p_j, H^{\otimes x} \otimes (T_{p_j}\mathcal{X})^{\otimes -i}) \to \oplus_{j=1}^s H^0(\hat{X}^{i+1}_{p_j}; H^{\otimes x}_{p_j,i+1}) \to \oplus_{j=1}^s H^0(\hat{X}^i_{p_j}; H^{\otimes x}_{p_j,i})$$

and the snake lemma, induces a canonical inclusion

$$\gamma^i_x : G^i_x / G^{i+1}_x \longrightarrow \oplus_{j=1}^s H^0(p_j, H^{\otimes x} \otimes (T_{p_j}\mathcal{X})^{\otimes -i}) \otimes K.$$

For every finite place $v \in M_{fin}$ both $(G^i_x/G^{i+1}_x)_v$ and $\oplus_j H^0(p_j, H^{\otimes x} \otimes (T_{p_j}\mathcal{X})^{\otimes -i})_v$ are equipped with norms, induced by the integral structure. Observe that, naturally the former has the sup norm. Thus we may compute the norm $\|\gamma^i_x\|_v$ of the operator $\gamma^i_x$.

When $f$ is an $LG$–germ, then one can bound the norms at the finite places of the $\gamma^i_x$'s.

**5.2 Theorem.** *With the notations as above, suppose that $f$ is a $LG$–germ of type $\alpha$ then there exists a constant $C$ such that*

$$\sum_{v \in M_{fin}} \log \|\gamma^i_x\|_v \leq [K : \mathbb{Q}]\alpha \cdot i \cdot \log(i) + C(i + x).$$

*Proof:* We first remark the following general statement: Let $k$ be a normed field and $\varphi : V^1_k \to V^2_k$ be a linear map between finite dimensional normed vector spaces over $k$. Let $V^i \subset V^i_k$ be the set of elements of norm less or equal then one. Suppose that there exists a constant $A \in k^*$ such that, for every element $v \in V^1$ we have that $\varphi(Av) \subset V^2$, then $\|\varphi\| \leq \frac{1}{\|A\|}$.

For every $j \in \{1, \ldots, s\}$, let $pr_j : \oplus_j H^0(p_j, H^{\otimes x} \otimes (T_{p_j}\mathcal{X})^{\otimes -i}) \to H^0(p_j, H^{\otimes x} \otimes (T_{p_j}\mathcal{X})^{\otimes -i})$ be the projection. Fix such a $p_j$. Let $v \notin S$. The restriction of $\hat{\mathcal{X}}_{p_j}$ to $\mathrm{Spec}(O_v)$ is isomorphic to $Spf(O_v[\![Z]\!])$ (via an isomorphism fixed as above).

Suppose that $P \in (G^i_x)_{O_v}$. Since we fixed an isomorphism of $E_{p_j}$ with the trivial vector bundle of rank $m$, the restriction of $P$ to $(\hat{\mathcal{X}}_{p_j})_v$ is represented by $(g_i, \ldots, g_m)$ with $g_i \in O_v[\![Z]\!]$. By definition $\langle P, f_j\rangle = \sum_{s=1}^m g_s \sum_\ell a_\ell(s) Z^\ell = h_j(Z)$. Since $P \in (G^i_x)_{O_v}$, we have that $h_j(Z) = \sum_{\ell=i}^\infty h_\ell Z^\ell$ and $pr_j \circ \gamma^i_x(P) = h_i$. Since $f_j$ is a $LG$–germ of type $\alpha$, we have that

$$\frac{\|i!\|_v^\alpha}{C^i_v} \cdot \|h_i\|_v \leq 1.$$

The norm on $\oplus_j H^0(p_j, H^{\otimes x} \otimes (T_{p_j}\mathcal{X})^{\otimes -i})$ is the sup of the norms on each factors, consequently, for every $v \notin S$ we have

$$\frac{\|i!\|_v^\alpha}{C^i_v} \cdot \|\gamma^i_x(P)\|_v \leq 1.$$

The conclusion follows from the remark at the beginning of this proof, the Stirling formula and the standard Cauchy inequality at the places in $S$.

## 6 $E$–sections of type $\alpha$.

In this section we will introduce the concept of $E$– sections of type $\alpha$ of a vector bundle over an affine curve. These are analytic sections of an algebraic vector bundle whose order of growth is the inverse of the type of their formal development in a fixed algebraic point. The main examples of $E$–sections are the $E$–functions of the theory of Siegel–Shidlowski (they are of type 1) or the more recent "arithmetic series of order $s$" introduced by André in [An1].

Let $\overline{X}_K$ be a smooth projective curve defined over a number field $K$. Fix $s$ points $p_1, \ldots, p_s \in \overline{X}(K)$.

Let $E_K$ be a vector bundle over $\overline{X}$ of rank $m$.

**6.1 Definition.** *A section $f := (f_1, \ldots, f_s) \in E_{p_1,K} \oplus E_{p_2,K} \oplus \ldots \oplus E_{p_s,K}$ is said to be a $E$–*section of type $\alpha$ *if the following holds:*

*a) For every $j$, the germ of section $f_i \in E_{p_j,K}$ is an LG–germ of type $\alpha$.*

*b) There exists a non empty subset $S_K \subseteq M_\infty$ of cardinality $a$ such that the following holds: for every $\sigma \in S$ there is an affine open subset $U_\sigma$ of $\overline{X}_\sigma$ and an* analytic *section $f_\sigma \in H^0(U_\sigma; E_\sigma)$ such that :*

*(b.1) For every $j$, the germ of $f_\sigma$ at $p_j$ is $f_j$;*

*(b.2) the section $f_\sigma$ has order of growth $\rho_\sigma = \frac{a \cdot s}{[K:\mathbb{Q}]\alpha}$.*

**6.2 Remark.** (a) An $E$–function in the sense of Siegel Shidlowski is an $E$– section of type 1; in this case we have only one point and $S_K = M_\infty$;

(b) An "arithmetic Gevrey series of order $s < 0$" in the sense of André [An1] is a $E$–section of type $-s$, again we only have one point and $S_K = M_\infty$.

(c) If $L/K$ is a finite extension and $f \in E_{p,K}$ is a $E$ section of type $\alpha$ over $K$, then $f \in E_{p,L}$ is a $E$–section of type $\alpha$; take as $S_L$ the set $\tau$ such that $\tau/\sigma$ for $\sigma \in S_K$.

(d) Notice that, on the projective line, the main differences between $E$–functions and $E$–sections are: (1) $E$ sections may have order of growth which is not one. (2) (more important) $E$–sections may have finitely many essential singularities, whereas $E$–functions are always entire functions.

(e) An interesting example of $E$ section, and our main theorem will concern such example, is given by an horizontal section of a fibre bundle with meromorphic connections having order of growth $\rho$ and assuming algebraic values at $\rho$ regular rational points: in each of the rational points the section will be an $LG$–germ of type 1.

(f) By Corollary 1.5 the order of growth cannot be less then $s$.

In this section we show that, given an $E$–section, it is possible to construct sections with high order of vanishing and bounded sup norm.

First of all we have to fix integral structures: As in the previous section, we suppose that $\mathcal{X} \to \mathrm{Spec}(O_K)$ is a regular projective model of $\overline{X}_K$ and $E_K$ extends to a vector bundle over $\mathcal{X}$. We suppose also that $H$ is a relatively ample line bundle on $\mathcal{X}$. For every place $\sigma \in M_\infty$, we suppose that $E_\sigma$ and $H_\sigma$ are equipped with smooth metrics (and the metric on $H$ is sufficiently positive). We also fix metrics on $(\overline{X})_\sigma$. Thus, for every integer $x$, the vector bundle $E^\vee(x)$ is an hermitian vector bundle over $\mathcal{X}$.

For every integer $x$, the $O_K$–module $H^0(\mathcal{X}, E^\vee(x))$ is equipped with a structure of *hermitian $O_K$–module*: for every $\sigma \in M_\infty$, $H^0(\overline{X}_\sigma; E^\vee(x)_\sigma)$ is equipped with the $L^2$ metric (notice that the $L^2$ norm and the sup norm are comparable by for instance [Bo] §4.1). As in the previous section, in the sequel, we will denote it by $G_x$.

Suppose we fixed an $E$–section $f \in \oplus_j E_{p_j,K}$ of type $\alpha$. Using the notations of the previous section, for every positive integer $i$, we obtain a natural $O_K$–linear map $G_x \to \oplus_j H^0(\hat{X}^i_{p_j}; H^{\otimes x}_{p_j,i}) \otimes K$. Again denote by $G^i_x$ its kernel. Put $c := \deg(H_K)$. We want to prove that, under these conditions, for every $\epsilon \in (0,1)$ there is a non vanishing section of bounded norm in $G_x^{x\frac{c}{s}m(1-\epsilon)}$.

**6.3 Theorem.** *Suppose that we fixed the hypotheses as above. Fix $\epsilon \in (0,1)$. Then, we can find a constants $a$, depending only on $\epsilon$, $\alpha$ the points $p_j$, but independent on the vector bundle $E$ and a constant $b$ for which the following holds: for every sufficiently big positive integer $x$ there is a* non zero *section $P \in G_x^{x\frac{c}{s}m(1-\epsilon)}$ such that*

$$\sup_{\sigma \in M_\infty} \{\log \|P\|_\sigma\} \leq a \cdot x \cdot \log(x) + b \cdot x.$$

Before we start the proof, we need to recall some classical tools form Arakelov geometry.

– Suppose that $E_1$ is an hermitian $O_K$–module and $L_1, \ldots, L_s$ are hermitian line bundles over $\mathcal{O}_K$. Let $\varphi : E_1 \to \oplus_j L_j \otimes K$ is an injective linear map. For every place $v \in M_K$ (finite or infinite) we denote by $\|\varphi\|_v$ the norm of $\varphi$. One easily find that, if $\varphi$ is non zero, then

$$\widehat{\deg}(E_1) \leq rk(E_1) \cdot \left( \sup_j \{\widehat{\deg}(L_j)\} + \sum_{v \in M_K} \log \|\varphi\|_v \right).$$

– There exists a constant $\chi(K)$ depending only on $K$ such that the following holds: Suppose that $E$ is an hermitian $O_K$–module. Suppose that $\widehat{\deg}(\overline{E}) \geq A$ then there exists a *non zero* element $x \in E$ such that

$$\sup_{\sigma \in M_\infty} \{\|x\|_\sigma\} \leq -\frac{A}{rk(E)} + \log(rk(E)) + \chi(K).$$

(cf. [BGS] thm. 5.2.4 and below it).

– if $x$ is sufficiently big, we may suppose that $\widehat{\deg}(G_x) \geq 0$.

We can now start the proof of the theorem.

*Proof:* As in the previous section, for every integer $i$, we have an injective map

$$\gamma_x^i : G_x^i / G_x^{i+1} \longrightarrow \oplus_{j=1}^r H^0(p, H^{\otimes x} \otimes (T_{p_j}\mathcal{X})^{\otimes -i}) \otimes K. \tag{6.3.1}$$

Remark that all the $G_x^i$'s are hermitian $O_K$ modules. From the properties listed above we find that we can find a constant $A$ depending only on $p$ and $H$ such that

$$\widehat{\deg}(G_x^{i+1}) \geq \widehat{\deg}(G_x^i) - rk(G_x^i/G_x^{i+1}) \left( A(x+i) + \sum_{v \in M_K} \log \|\gamma_x^i\|_v \right).$$

Since $f$ is an $E$–germ of type $\alpha$, by theorem 5.2, there is a constant $c$ such that

$$\sum_{v \in M_{fin}} \log \|\gamma_x^i\|_v \leq [K:\mathbb{Q}]\alpha \cdot i \cdot \log(i) + c \cdot (i+x).$$

Let $S_K$ be the set of infinite places involved in the definition of $E$–germ. By the classical Cauchy inequality there is a constant $C$ such that, if $\sigma$ is an infinite place not contained in $S_K$, then

$$\log \|\gamma_x^i\|_\sigma \leq C(x+i).$$

In order to estimate the norm in the places of $S_K$ we need a refinement of Theorem 4.9.

Let $\sigma \in S_K$. Let $j \in \{1, \ldots, s\}$, we put on the line bundle $\mathcal{O}_{U_\sigma}(p_j)$ the following metric: Suppose that $\mathbb{I}_{p_j}$ is the canonical section of $\mathcal{O}_{U_\sigma}(p_j)$, then we define $\|\mathbb{I}_{p_j}\|(z) = \exp(\frac{1}{2} g_{p_j}(z))$. By adjunction, this defines a norm on $T_{p_j}\overline{X}_\sigma$. Let $s \in (G_i)_\sigma$; then $f_\sigma(s) \cdot \prod_j \mathbb{I}_{p_j}^{-i}$ is an holomorphic section $\tilde{F}$ of $(H^{\otimes x}(-\sum_j ip_j))_\sigma$. To compute the norm of $\gamma_x^i$ at the places in $S_K$ we have to compare $\|\tilde{F}\|(p_j)$, for every $j$, with $\|s\|_\infty$.

By Stokes formula we find, for every real number $r$,

$$\int_{U_\sigma} \log \|\tilde{F}\| \cdot dd^c g_p^r = \int_{U_\sigma} dd^c \log \|\tilde{F}\| \cdot g_p^r.$$

By proposition 4.3 as soon as $r \gg 0$ we may suppose that, if $g_{p_j}(z) \geq r$, then $g_{p_{j_i}}(z) = g_{p_j}(z)(1+\epsilon(z))$ with $|\epsilon(z)| \leq \epsilon$ Thus, by the property of the Green function recalled in §3, the definition of the norm on $\mathcal{O}(p_j)$, the Cauchy–Schwarz inequality and the Poincaré–Lelong formula we find

$$\log \|s\|_\infty + \int_{S(r)} \log \|f_\sigma\| d^c g_p - is(1-\epsilon)\log(r) - \log \|\tilde{F}\|(p) \geq -x(H; U_\sigma)(r).$$

Thus, we can find constant $C$ and $\epsilon > 0$, depending only on $H$ and a constant $\lambda_\sigma$ depending on $f$, such that, as soon as $r \gg 0$

$$\log \|\gamma_x^i\|_\sigma \leq -is(1-\epsilon)\log(r) + x \cdot C \log(r) + \lambda_\sigma r^{\frac{as}{[K:\mathbb{Q}]\alpha}(1+\epsilon)}.$$

For each $i$ we put $r = i^{\frac{\alpha[K:\mathbb{Q}]}{as(1+\epsilon)}}$ and we obtain that there are constants $C_1$ $C_2$ and $\epsilon_1$ with $C_1$ depending only on $\alpha$ and $H$, and $\epsilon_1$ as small as we want, in particular independend on $E$, $i$ and $x$ such that

$$\sum_{v\in M_K} \log \|\gamma_x^i\|_v \leq xC_1 \log(i) + \epsilon_1 i \log(i) + C_2(i+x).$$

Observe that $rk(G_x^i/G_x^{i+1}) \leq s$ consequently we can find constants $C_i$ with $C_4$ depending only on $H$, $s$ and $\alpha$ such that, summing up all together we obtain

$$\widehat{\deg}(G_x^{x\frac{c}{s}m(1-\epsilon)}) \geq C_3x^2 - C_4 \left( \sum_{i=1}^{x\frac{c}{s}m(1-\epsilon)} x\log(i) + \epsilon_1 i \log(i) + C_2(i+x) \right).$$

Thus, since we can take $\epsilon_1$ very small compared to $m^2$, there are constants $C_6$ and $C_7$ with $C_6$ independent on $E$ and $C_7$ depending on $m$, $f$ and $H$ such that

$$\widehat{\deg}(G_x^{x\frac{c}{s}m(1-\epsilon)}) \geq -\left(C_6 m \cdot x^2 \cdot \log(x) + C_7 \cdot x^2\right).$$

By Riemann–Roch theorem, $rk(G_x)$ is about $cmx$. By the filtration 6.3.1, the rank of $G_x^{x\frac{c}{s}m(1-\epsilon)}$ is bigger then $\epsilon mx$. Consequently, there is a *non zero section* $P$ of $G_x^{x\frac{c}{s}m(1-\epsilon)}$ such that

$$\sup_{\sigma\in M_\infty} \{\log \|P\|_\sigma\} \leq \frac{mC_8}{m\epsilon} \cdot x \cdot \log(x) + C \cdot x.$$

The conclusion follows since, $C_8$ depend only on the $p_j$, $H$ and it is independent on $E$.

We suppose that we fixed $s$ rational points and a fibre bundle $(E,\nabla)$ with a meromorphic connection. We suppose that we fixed an $E$ section of type $\alpha$ in the in the neighborhood of these points which is horizontal for the connection. Using the connection, we may now take derivatives of $P$ in order to construct other sections with the same properties.

First of all we have to assure that the derivative of an integral section is again an integral section: we may extend the connection $\nabla : E^\vee(x)_K \to E^\vee(x)_K \otimes \Omega^1_{\hat{X}}(D_K)$ to an *integral connection*:

$$\nabla : E(x) \to E \otimes \omega_{\mathcal{X}/O_K}(D+V)$$

where $V$ is a vertical divisor. We also fix an integral element $\partial \in T_{\mathcal{X}/O_K}(D)$ which, generically, do not vanish at the points $p_i$'s, in case we may suppose the degree of $D$ very big. By construction, if $P \in H^0(\mathcal{X}, E^\vee(x))$, then $\nabla_\partial(P) \in H^0(\mathcal{X}, E^\vee(x+2+V))$; in particular notice that $\nabla_\partial(P)$ is an integral element.For every point $p_j$, the order of vanishing of $\langle\nabla_\partial(P), f_j\rangle$in $p$ is one less then the order of vanishing in $p$ of $\langle P, f\rangle$ in $p$. Moreover a straightforward application of the classical Cauchy–Schwarz inequality implies that, for every complex embedding $\sigma$, the linear map $(\nabla_\partial)_\sigma : E^\vee(x)_\sigma \to (E^\vee_\sigma(x+2))_\sigma$ has bounded norm. Thus we proved:

**6.4 Proposition.** *There is a constant $A$ depending only on $(E,\nabla)$ and $\partial$ such that the*

*following holds: if $P \in H^0(\mathcal{X}, E^\vee(x))$ is an integral section such that $\sup_\sigma\{\log \|P\|_\sigma\} \leq C$ and, for every $j$ we have $ord_{p_j}(\langle P, f_j\rangle) \geq C_1$ then:*

*(i) $\nabla_\partial(P)$ is an integral section of $E^\vee(x+2+V)$ such that*

$$\sup_\sigma\{\log \|\nabla_\partial(P)\|_\sigma\} \leq C + A;$$

*(ii) For every $j$ we have $ord_{p_j}(\langle \nabla_\partial(P), f_j\rangle) \geq C_1 - 1$.*

## 7 Proof of the main theorem.

In this section we will show how to generalize the Siegel Shidlovski theory to an arbitrary curve and to a connection with arbitrary meromorphic singularities and $E$ sections over an arbitrary set of points.

**7.1 Theorem.** *Let $\overline{X}/_K$ be a smooth projective curve defined over the number field $K$. Let $D_K$ be an effective divisor on $\overline{X}$ and $(E_K, \nabla_K)$ be a fibre bundle of rank $m > 1$ with connection with meromorphic poles along $D_K$. Let $p_1, \ldots, p_s \in \overline{X}(K)$ be rational points.*

*Let $f := (f_1, \ldots, f_s) \in \oplus_j E_{p_j,K}$ be an* horizontal section *which is an $E$–section of type $\alpha$.*

*Let $S_K$ be the subset of infinite places of $K$ involved in the definition of $f$ and $\sigma \in S_K$.*

*Suppose that $q \in \overline{X} \setminus \{D, p_1, \ldots, p_s\}$. Then*

$$Trdeg_K(K(f_\sigma(q))) = m.$$

**7.2 Remark.** If we apply the theorem to $\mathbb{P}^1$ with $s = 1$ and we suppose that the horizontal section is an $E$ function, we find the classical theorem of Siegel and Shidlovski cf. [La].

*Proof:* We fix models $\mathcal{X}$ of $\overline{X}$, $D$ of $D_K$ and $(E, \nabla)$ of $(E_K, \nabla_K)$. We also fix a positive metric on the ample line $H := \mathcal{O}(D)$. Let $c$ be the degree of $H_K$, adding some points to $D$ if necessary we may suppose that $c > s$. We eventually fix an integral derivation $\partial \in H^0(\mathcal{X}, (\omega^1_{\mathcal{X}/O_K})^\vee(D))$ which do not vanish at the points $p_j$'s and $q$; notice that this can be done once we suppose that $c$ is big compared to $s$.

We want to apply 2.4, thus we need $m$ linearly independent sections of $E^\vee(x)_q$ with satisfying the hypotheses of loc. cit.

By Theorem 6.3, for every $x \gg 0$ we may construct $P_1 \in H^0(\mathcal{X}; E^\vee(x))$ such that, using the notations of loc. cit., for every $j$ we have $ord_{p_j}(F_1) \geq x\frac{c}{s}m(1-\epsilon)$ and $\sup_\sigma\{\log \|P_1\|_\sigma\} \leq ax\log(x) + c_1x$.

Let $P_i := \nabla_\partial P_{i-1}$. Applying proposition 6.4, we construct then $m$ integral sections $P_1, \ldots, P_m$ such that (again using notations of loc. cit.) $\sup_\sigma\{\log \|P_i\|_\sigma\} \leq ax\log(x) + c_2x$ and $ord_p(F_i) \geq x\frac{c}{s}m(1-\epsilon) - m$.

Since we suppose that $c > s$ and $x \gg 0$, we may apply The Zero Lemma 3.4 and we obtain that $P_1, \ldots, P_m$ are linearly independent over $K(\overline{X})$. Observe that $P_i \in H^0(\mathcal{X}, E^\vee(x + 2m + V))$ for some fixed vertical divisor $V$. Consequently there is a constant $c_3$ such that

$$\deg(P_1 \wedge \ldots \wedge P_m) = mcx + c_3.$$

By Cramer rule,

$$(P_1 \wedge \ldots \wedge P_m) \otimes f = \sum_i (-1)^i (P_1 \wedge \ldots \wedge \hat{P}_i \wedge \ldots \wedge P_m) \otimes F_i;$$

thus, for every $j$, $ord_{p_j}(P_1 \wedge \ldots \wedge P_m) \geq x\frac{c}{s}m(1-\epsilon) - m$; consequently, constants $\epsilon_1$ and $c_4$ such that

$$ord_q(P_1 \wedge \ldots \wedge P_m) \leq cx\epsilon_1 + c_4.$$

**7.3 Lemma.** *With the notations as above, there are constants $c_i$'s independent on $E$ (in particular independent on $m$) and constants $b_j$ for which the following holds: for every $x \gg 0$ there exist $m$ indices $\ell_1 < \ell_2 < \ldots < \ell_m$ with $\ell_m \leq m + c_1 x + c_2$ such that:*

*(a) $P_{\ell_1} \wedge \ldots \wedge P_{\ell_m}$ is an integral section* non vanishing at *$q$;*

*(b) $\sup_\sigma \{\log \|P_{\ell_i}\|_\sigma\} \leq c_3 x \log(x) + b_1$;*

*(c) For every $j$ we have $ord_{p_j}(F_{\ell_i}) \geq c_4 x(m-1) - b_2$.*

The Lemma implies the theorem: indeed, by theorem 4.9, the $P_{\ell_i}$ verify the hypotheses of proposition 2.4.

*Proof:* (of lemma 7.3) The only thing we have to prove is (a); indeed (b) and (c) are consequence of proposition 6.4.

Let $\alpha$ be the order of $(P_1 \wedge \ldots \wedge P_m)$ at $q$. We know that $\alpha \leq cx\epsilon + c_4$. By induction we have that, if $h_1 < \ldots < h_m$,

$$\nabla_\partial(P_{h_1} \wedge \ldots \wedge P_{h_m}) = \sum_{s_1 < \ldots < s_m} a_s \cdot (P_{s_1} \wedge \ldots \wedge P_{s_m})$$

with $s_i \leq h_m + 1$ and suitable universal constants $a_s$. Consequently, denoting by $\nabla_\partial^{\circ\alpha}(\cdot)$ the operator $\nabla_\partial \circ \ldots \nabla_\partial(\cdot)$ ($\alpha$ times), we have that

$$0 \neq \nabla_\partial^{\circ\alpha}(P_1 \wedge \ldots \wedge P_m)|_q = \sum_{s_1 < \ldots < s_m} a_s \cdot (P_{s_1} \wedge \ldots \wedge P_{s_m})|_q$$

with $s_m \leq m + cx + a$. Thus there exists $\ell_1 < l_2 < \ldots < \ell_m \leq m + cx + a$ such that

$$(P_{\ell_1} \wedge \ldots \wedge P_{\ell_m})|_q \neq 0.$$

The conclusion follows.

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